# On the Use of a Relaxation Oscillator as a Characterized Voltage-to-Frequency Converter*

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**Summary—This paper considers the circuit synthesis of a particular type of voltage-to-frequency converter with a predetermined, but arbitrary, voltage-to-frequency characteristic. The basis of this converter is a relaxation oscillator consisting of a linear energy storage network, a combined signal and bias voltage source which provides power to the oscillator, and a single four-layer diode or similar relaxation switch.**

**The paper first attempts to show what forms of characterization such a circuit might reasonably be expected to give in a voltage-to-frequency conversion. A more rigorous argument follows where it is shown that the voltage waveform out of such a network is a linear function of both the signal voltage and the diode current, the current being treated as an impulse discharge from some effective capacity. The sum equation, thus derived, can be solved using a remarkable theorem about the Mobius function of number theory to give the exact form of the desired impulse response of the linear network. As an example of the application and practical limitations of the approach, a network for a fractional root converter is derived and discussed in terms of a completed square-root converter.**

## INTRODUCTION

The purpose of this paper is to consider a relaxation oscillator for use as a voltage-to-frequency converter with an arbitrarily characterized voltage-to-frequency relationship. All other things being equal, such an approach offers the advantage of characterization to any degree of accuracy with a single, highly accurate, and easily realized nonlinearity, the relaxation switching mechanism. The principle result is an expression for the voltage-to-frequency relationship of the oscillator limit cycle in terms of the linear impedance functions in the circuit. Inverting the expression makes possible synthesis of a circuit with the desired limit cycle characteristic.

The simplest oscillator of the type to be considered is the RC, four-layer diode relaxation oscillator shown schematically in Fig. 1. In this circuit, the diode acts as a switch, shorting when output voltage ($V_o$) exceeds firing voltage ($V_f$) and opening upon discharge of the capacity (C). If $V_i$, a dc input voltage, is much larger than $V_f$, the output voltage will follow the linear sawtoothed form expected of such a circuit. Since $V_o$ will then have an average value of $V_f/2$, the frequency of the device will be proportional to $(V_i - V_f/2)$. Also, if $V_i$ is less than $V_f$, no oscillation can occur. Assuming perfect shorting the circuit behavior is defined by the equation

$$V_o = V_i(1 - e^{-t/(RC)})$$

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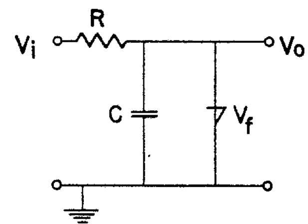


Fig. 1—RC, four-layer diode, relaxation oscillator.

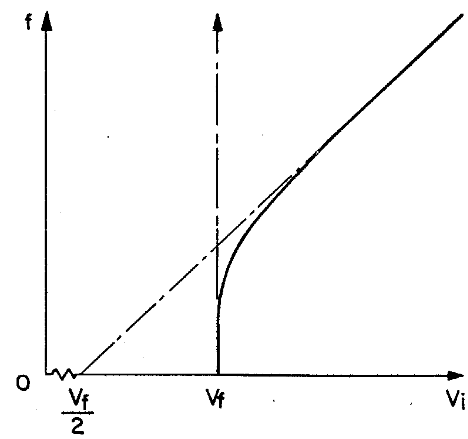


Fig. 2—Voltage-to-frequency characteristic of the RC relaxation oscillator.

If diode firing is considered to occur at a time ($t$) equal to one period of oscillation ($T$), the equation at the time of firing becomes

$$V_f = V_i(1 - e^{-T/(RC)})$$

$$= V_i(1 - e^{-1/(fRC)}) ,$$

where the frequency of oscillation ($f$) is defined as $1/T$. Solving for frequency

$$f = \frac{-1}{RC \ln\left(1 - \frac{V_f}{V_i}\right)}$$

or in the limit (see Fig. 2) as $V_i/V_f \rightarrow \infty$,

$$f = \frac{1}{RCV_f}\left(V_i - \frac{V_f}{2}\right).$$

Intuitively, one might reasonably use this expression with appropriate bias on input voltage to approximate any of a series of logarithmic or fractional power law functions which are of this general appearance. If $V_i$ is kept small, the linear portion of the curve need not be considered as a limitation.

## A More General Case

The question arises as to whether a more complicated RC or RLC network might improve the approximation to any given function. The more general circuit shown in Fig. 3 includes a four-layer diode or similar switching system, which is meaningful only if it is allowed to discharge a capacitor $C$. In other words, the network must have an output-impedance-impulse response with a finite nonzero initial value. Because of this initial discontinuity, the circuit will exhibit the same saw-toothed waveform and linear behavior to large input voltages as the RC oscillator, and will oscillate only when the input exceeds some positive value.

Consider the general two-port network [1] shown in Fig. 4, defined by the equations

$$V_i = Z_{ii} \bullet I_i + Z_{io} \bullet I_o \qquad (1)$$
$$V_o = Z_{oi} \bullet I_i + Z_{oo} \bullet I_o .$$

The dot operator (•) indicates either multiplication of Laplace transforms or convolution of time functions. These equations, rearranged in terms of mixed parameters, give

$$V_o = Z_{oi} \bullet \left(\frac{V_i - Z_{io} \bullet I_o}{Z_{ii}}\right) + Z_{oo} \bullet I_o$$
$$= \left(Z_{oo} - \frac{Z_{io} \bullet Z_{oi}}{Z_{ii}}\right) \bullet I_o + \frac{Z_{oi}}{Z_{ii}} \bullet V_i ,$$

which may be redefined

$$V_o \equiv Z_o \bullet I_o + K \bullet V_i . \qquad (1a)$$

If $K$ and $Z_o$ are treated as Laplace transforms, $K$(s) and $Z_o$(s), then $K$ is a transfer function and $Z_o$(s) is the output impedance with the input short-circuited. Alternatively, if $Z_o$ is treated as a time function, then $Z_o$(t) is the voltage response to a unit current impulse applied at the output with the input short circuited.

Consider the limit cycle behavior of the circuit in Fig. 3 under a steady dc input voltage ($V_i$), treating the diode as the current source $I_o$ of Fig. 4. The value of $K \bullet V_i$ will have settled to some value $kV_i$ where $k$ is a constant which may be taken as unity without loss of generality. Expressed another way, if a constant voltage $V_i$ is applied to the input and the diode removed, $V_o$ will settle to a value $kV_i$ or $V_i$. For this reason the steady-state analysis will not involve $K$. Thus $K$ may be independently determined for optimum transient behavior.

At time ($t_n$) when the capacitor discharges through the diode, it loses an amount of charge equal to $CV_f$. Thus the current through the diode can be represented by a train of impulses of form $-CV_f\delta(t - t_n)$ where $\delta(t)$ is the impulse function and $t_n$ is the starting time of the $n$th impulse. Since future impulses can have no effect on the present, they can be disregarded; $t_n$ will always be negative. Assuming this impulse train has a steady frequency ($f$) or period ($T$), if $t_o = 0$ and the impulses are ordered in $n$, then $t_n = -nT$. The current through the diode will

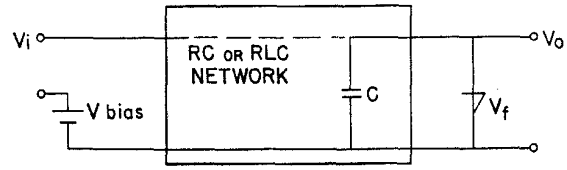


Fig. 3—General RC or RLC, four-layer diode, relaxation oscillator.

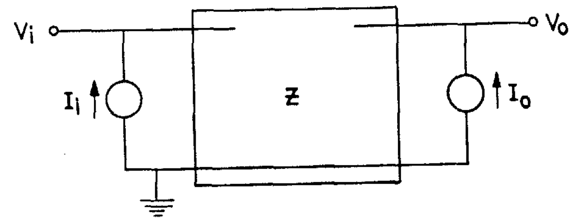


Fig. 4—General two-port linear network.

therefore be a sum of impulses

$$I_o(t) = -\sum_{n=0}^{\infty} CV_f\delta(t+nT) .$$

Since $V_o = Z_o \bullet I_o + K \bullet V_i$ (1a) the limit cycle behavior for unity transfer function follows:

$$V_o = -Z_o(t) \bullet \sum_{n=0}^{\infty} [CV_f\delta(t+nT)] + V_i .$$

By the definition of an impulse response,

$$Z_o(t) \bullet \delta(t+nT) = Z_o(t+nT) .$$

Therefore

$$V_o = \sum_{n=0}^{\infty} [CV_fZ_o(t+nT)] + V_i . \qquad (2)$$

This is shown pictorially in Fig. 5 where $Z_0(t)$ is assumed to be a continuous function except at $t = 0$. An input signal ($e_s$) may be defined such that $e_s(f)$ is the inverse of the desired characteristic. If a bias equal to $V_f$ is required, then $e_s = V_i - V_f$. At $t = 0^+$, immediately following the discharge, $V_o = 0$ and $CV_f Z_o(0^+) = V_f$. This gives from (2),

$$e_s\left(\frac{1}{T}\right) = \sum_{n=1}^{\infty} CV_fZ_o(nT) . \qquad (3)$$

A general inversion to this equation, when such an inversion is possible, exists in the number theory [2]

$$Z_o(T) = \frac{1}{CV_f}\sum_{n=1}^{\infty} \mu(n)e_s\left(\frac{1}{nT}\right) . \qquad (4)$$

The Mobius function $\mu(n)$ depends on the number of prime factors of $n$. If any prime factor of $n$ occurs more than once in the factorization, $\mu(n) = 0$. Otherwise, $\mu(n) = (-1)^p$ where $p$ is the number of prime factors (excluding 1, if it is treated as a prime). That is, $\mu(1) = 1$.

By linearity, the sum of two solutions to (3) is also a solution. In principle, any physically realizable $Z_o(t)$, as defined by (4), can be synthesized by standard procedures [3]. It is also possible to establish conditions that must

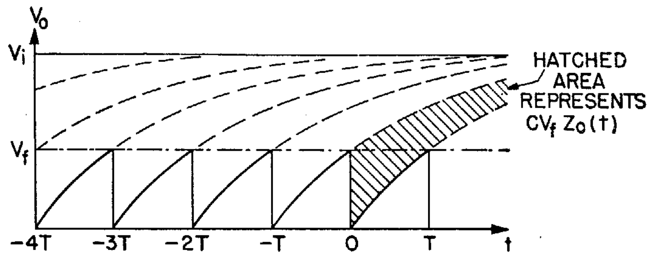


Fig. 5—Output voltage of the general four-layer diode oscillator.

be met if (4) is to represent the behavior of actual networks. Basically, these are equivalent to the requirement that the output voltage ($V_o$) can never equal or exceed $V_f$ at any time between diode firings. One sufficient condition is that $Z_o(t)$, in addition to being positive real [3], must be monotonic decreasing in $t$. Use of $Z_o(t)$ not of this form may lead to some interesting discontinuous behavior which requires an extension of the analysis.

Explicit solutions to (3) do exist for certain forms of $e_s(1/T)$. For instance,

$$e_s\left(\frac{1}{T}\right) = \frac{e^{-T/\tau}}{1+e^{-T/\tau}} = \sum_{n=1}^{\infty} e^{-(nT)/\tau}$$

gives

$$Z_o(nT) = \frac{e^{-(nT)/\tau}}{CV_f}.$$

This expression, with appropriate constants and change of variable, describes the RC oscillator of the Introduction.

## Inverse Power Law Characterization

A more interesting example, which shows some of the properties of this method of characterization, is based on an even simpler inversion of (3). If $f = f(e_s)$ is of the form $e_s^{1/p}$, $e_s(1/T) = 1/T^p$, so that the relation

$$\sum_{n=1}^{\infty} \frac{1}{(nT)^p} = \frac{1}{T^p}\sum_{n=1}^{\infty}\frac{1}{n^p} \tag{5}$$

implies that

$$Z_o(nT) = \frac{1}{CV_f\zeta(p)} \cdot \frac{1}{(nT)^p} \ . \tag{5a}$$

In this equation $\zeta(p) = \sum_{n=1}^{\infty} 1/n^p$ is the Riemann zeta function of the number theory which is tabulated in standard tables [5]. It is tempting to object, at this point, that no such impulse function is realizable. However, it is possible to design a network for which (5a) holds for all values of $T$ larger than some particular value; that is, in effect, requiring that $e_s(f) = f^p$ for any value between zero and some maximum $f$. This is all that can be expected from any physical device.

The initial work has been done using an RC network for both practical and mathematical reasons. For such a network, with a finite number of lumped elements, $Z_o(t)$ can be expanded as a sum of exponentials [3],

$$Z_o(nT) = \sum_{n=1}^{N} f_n e^{-x_n t} \ , \tag{6}$$

$f_n$ and $x_n$ being network parameters.

It is not unreasonable to generalize this sum to an integral, giving

$$Z_o(t) = \int_0^{\infty} f(x) e^{-xt} dx \ . \tag{6a}$$

The effect of this generalization is to include, as legitimate networks, distributed parameter networks and networks with an infinite number of lumped elements.

Noting the similarity of this expression to the expression for the Laplace transform, it is apparent from the tabulated expression [5] for $\mathcal{L}[t^{p-1}]$ that if by (5a)

$$Z(t) = \frac{1}{CV_f\zeta(p)t^p} \ ,$$

then in (6a)

$$f(x) = \frac{x^{p-1}}{CV_f\zeta(p)(p-1)!} \ .$$

As before mentioned, the impulse response of (5a) cannot be realized, but the related form

$$Z_o(t) = \frac{1}{CV_f\zeta(p)}\int_0^a \frac{x^{p-1}}{(p-1)!} e^{-xt} dx \tag{6b}$$

has a finite value for the expression

$$Z_o(0^+) = \frac{1}{CV_f\zeta(p)}\int_0^a \frac{x^{p-1}}{(p-1)!} dx = \frac{a^p}{CV_f\zeta(p)p!} \ , \tag{6c}$$

and is therefore realizable. Since by definition $Z_o(0^+) = 1/C$, then $a^p = V_f\zeta(p)\, p!$. For $p$ an integer; (6b) can be integrated [5] to give

$$Z_o(t) = \frac{1}{CV_f\,\zeta(p)\,t^p}\left[1 - e^{-at}\sum_{n=0}^{p-1}\frac{(at)^n}{n!}\right] \ , \tag{6d}$$

and the difference between the desired form of $Z_o(t)$ as in (5a) and the approximation of (6b) becomes

$$\Delta Z_o(t) = \frac{1}{CV_f\,\zeta(p)\,t^p} - Z_o(t)$$

$$= \frac{e^{-at}}{CV_f\,\zeta(p)\,t^p}\sum_{n=0}^{p-1}\frac{(at)^n}{n!} \ .$$

which can be made arbitrarily small compared to $1/[CV_f\,\zeta(p)\,t^p]$ as t gets large. Therefore, by (3) with $t = nT$ replaced by $m/f$, the corresponding error in characterization has the form

$$\Delta e_s(f) = \sum_{m=1}^{\infty} CV_f \Delta Z_o\left(\frac{m}{f}\right)$$

$$= \sum_{m=1}^{\infty} \frac{e^{-a(m/f)}}{CV_f\zeta(p)}\left(\frac{f}{m}\right)^p \cdot \sum_{n=0}^{p-1}\frac{1}{n!}\left(\frac{am}{f}\right)^n \ .$$

This can be made arbitrarily small compared to $e_s(f)$ if the maximum frequency is itself made small enough.

It remains to express a network in terms of some lumped element expansion. The impedance function corresponding to $Z_o(t)$ is the Laplace transform of $Z_o(t)$.

By (6b)

$$Z_o(s) = \mathcal{L}[Z_o(t)] = \mathcal{L}\left[\frac{1}{CV_f\zeta(p)}\int_0^a \frac{x^{p-1}}{(p-1)!}e^{-xt}dx\right]$$

or, interchanging integration and the Laplace transformation,

$$Z_o(s) = \frac{1}{CV_f\zeta(p)(p-1)!}\int_0^a \frac{x^{p-1}}{(s+x)}dx \ . \tag{8}$$

For $p$ an integer with $\zeta(p)(p-1)!$ equal to $a^p/pV_f$ as in (6c), (8) is integrated [5] to give

$$Z_o(s) = \frac{p}{Ca^p}\left[\sum_{m=1}^{p-1}\frac{a^m}{m}(-s)^{p-m-1} + (-s)^{p-1}\ln\left(1+\frac{a}{s}\right)\right] , \tag{8a}$$

which becomes, with $\ln(1+a/s)$ equal to the series $-\sum_{m=1}^{\infty}(1/m)(-a/s)^m$ [5],

$$Z_o(s) = \frac{-p\left(-\frac{s}{a}\right)^{p-1}}{Ca}\left[\sum_{m=p}^{\infty}\frac{1}{m}\left(-\frac{a}{s}\right)^m\right] . \tag{8b}$$

Defining $R$ as the resistive component of $Z_o(s)$ as $s \to 0$, and recognizing that $s\ln(1+a/s) \to 0$ as $s \to 0$, we obtain from (8a)

$$R = \frac{p}{Ca(p-1)}$$

and if

$$\tau = RC$$

then,

$$a = \frac{p}{\tau(p-1)}$$

and (8b) becomes

$$Z_o(s) = -R(p-1)\sum_{m=p}^{\infty}\frac{1}{m}\left[-\frac{p}{p-1}\frac{1}{\tau s}\right]^{m-p+1} . \tag{8c}$$

This power series may be expanded in a continued fraction and realized as a ladder network. For instance, for $p = 2$, $\zeta(p)$ = „$^2$/6 = 1.64 and

$$Z_o(s) = -R\sum_{m=p}^{\infty}\frac{1}{m}\left(\frac{-2}{\tau s}\right)^{m-1} . \tag{9}$$

One possible expansion which may be obtained by standard methods [3], [4] has the form

$$Z_o(s) = \frac{1}{\frac{1}{R}+Cs+}\,\frac{1}{3R+}\,\frac{1}{\frac{Cs}{2}+}\,\frac{1}{5R+}\cdots\frac{1}{\frac{Cs}{n}+}\,\frac{1}{(2n+1)R+}\cdots , \tag{9a}$$

and has the ladder realization of Fig. 6. Because of the difficulty of working back and forth between Laplace transforms and time domain characterization, studies on the effects of truncating the ladder expansion or on the effects of error in components are best handled by experiment or digital simulation. As would be expected, such studies have indicated that the accuracy required of a component depends on its distance from the diode and decreases markedly with this distance. Similar difficulty hinders detailed study of transient behavior but experiment indicates that this does not create any unexpected problems. The above analysis has also failed to consider the effect of noise which causes appreciable short-term frequency variations, but which does not adversely affect the long-term frequency behavior.

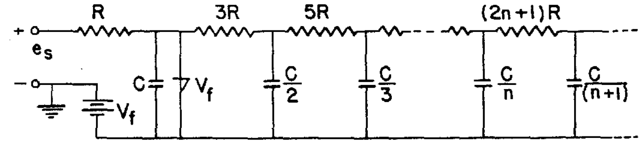


Fig. 6—A ladder realization of an oscillator with square-root characterization.

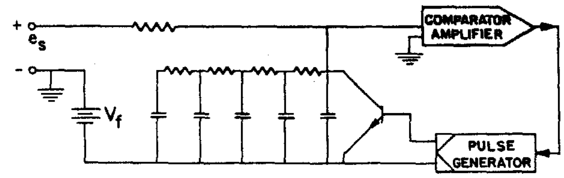


Fig. 7—A modified relaxation oscillator.

The author has built and tested a square-root characterized oscillator with ten RC stages in the ladder network which has a maximum long-term measured error in output frequency of 0.2 per cent of the maximum output frequency in agreement with theory. Although the detail of the error calculations preclude presentation here, experimental and computer studies indicate that for this particular circuit the allowed error in components almost doubles with each succeeding RC stage. Since system error is roughly in proportion to allowed component error this last statement is roughly equivalent to saying that per cent error varies as $1/2^n$ with n the number of stages. In the circuit design of this oscillator (Fig. 7), the diode has been replaced by a transistor switching circuit to minimize error caused by the separation of bias and firing voltages, each determined by a different device. For this circuit and an RC network, the maximum value for $e_s$, consistent with 0.2 per cent accuracy as calculated from (7) is of the order of 0.1 $V_f$, making the variation allowed between bias and firing voltage extremely critical in terms of per cent accuracy when the original circuit design of Fig. 6 is used.

## Discussion

Practical considerations of this kind suggest that a simpler and improved circuit might be realized if $e_s$ could be larger in comparison to $V_f$. Since this maximum input signal corresponds to the maximum output frequency discussed before, it is related to the network approximation used to obtain the impulse response.

The RC approximation used has an impulse response which approaches arbitrarily close to $1/[CV_f\zeta(p)\,t^p]$ as $t \to \infty$ as seen in Fig. 8(a) but does not ever exactly equal it. From very elementary considerations of physical realizability and conservation of energy, we can never obtain a $Z_o(t)$ where $Z_o(t) > Z_o(0^+) = 1/C$. That is, the energy applied by a single impulse to the capacitor $C$ and the corresponding voltage $Z_o(t)$ can never exceed their initial values. Therefore, the best conceivable impulse response, from this point of view, would be the function defined by

$$Z_o(t) = \frac{1}{C} \quad \text{for} \quad t < \left(\frac{1}{V_f\zeta(p)}\right)^{1/p},$$

and otherwise by

$$Z_o(t) = \frac{1}{CV_f\zeta(p)t^p}$$

as shown in Fig. 8(b). For

$$T_{min} = \left(\frac{1}{V_f\zeta(p)}\right)^{1/p},$$

$$e_s max = e_s\left(\frac{1}{T_{min}}\right) = \frac{1}{(T_{min})^p} = V_f\zeta(p) \ .$$

While such *a* $Z_o(t)$ does not have a positive real transform and therefore is not realizable, it does provide a maximum limit to what may be done. The difficulty in a general approach to the problem is the lack of any convenient general expansion for an RLC network in terms of $Z_o(t)$. However, it is possible to expand the impedance function in terms of classes of positive real functions. One such class of functions, which has provided constructive results, is the class whose impulse responses *Z(t)* are right triangles with the right angle at the vertex as shown in Fig. 8(c). If these are used as a basis to expand in integrations similar to (6) and (8a), it is possible to show that a network exactly equal to $1/[CV_f\zeta(p)\,t^p]$ can be physically realized for

$$t > \left((p+1)\frac{1}{V_f\zeta(p)}\right)^{1/p}$$

and

$$e_s < \frac{V_f\zeta(p)}{p+1}$$

[See Fig. 8(d).] In fact, better networks may be obtained. Once the corresponding positive real $Z_o(t)$ is obtained, standard RLC synthesis techniques are applicable. The above results are not appreciably changed by the introduction of scaling constants. In particular, the limits on $e_s$ for a network are completely unaffected by such a change.

1t is clearly possible to extend this type of analysis to oscillators in which added flexibility is obtained by shorting more than one capacitor. This more general case will give rise to simultaneous sum equations of the type considered. These equations are soluble by the same inversions, even in the more general case.

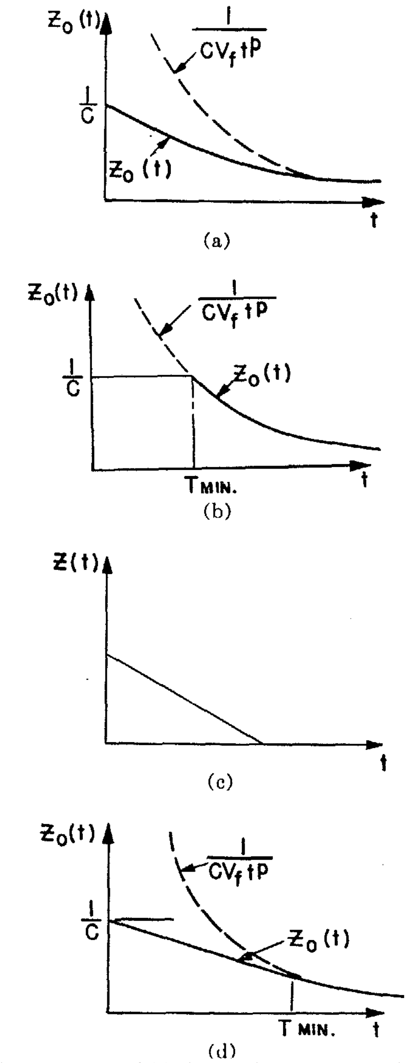


Fig.8— Impulse response of (a) the RC inverse power law characteristic network, (b) an idealized nonrealizable inverse power law characteristic network, (r.) a class of realizable building block networks, (d) the inverse power law characteristic network realized as an integral of these building blocks.

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